Compact Coadjoint Orbits

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Abstract

I give an answer to the question "Which groups have compact coadjoint orbits?". Whilst I thought that the answer, which is straightforward, must be in the literature, I was unable to find it. This note aims to rectify this.

It is also a plea: If the result is already published then I would like to be told the reference.

1 Introduction

This note is a response to the question of "Which connected Lie groups have compact coadjoint orbits?" which I was asked whilst on a visit to the Differential Geometry group at the ULB in Brussels. The obvious answer is that they are the coadjoint orbits of a compact Lie group. But that is not completely true as the origin in $\mathfrak{g}^*$ is a compact coadjoint orbit in every Lie algebra. More generally, an element of the dual of a Lie algebra which vanishes on the derived algebra will be a fixed point of the coadjoint action and so give an orbit consisting of a single point.

We aim to show by elementary means that these two cases essentially account for all compact orbits in the following sense: if $\mathcal{O}$ is a compact coadjoint orbit for the group

G then there is a closed normal subgroup H of G which fixes each point of $\mathcal{O}$, and such that G/H is a compact semisimple Lie group in such a way that $\mathcal{O}$ is the sum of a G-fixed element of $\mathfrak{h}^*$ and a coadjoint orbit of G/H pulled back to $\mathfrak{g}^*$.

I was surprised that I could not find a reference for such a basic result despite the length of time for which coadjoint orbits have been studied. This note is an attempt to give an elementary answer to the question, and at the same time an appeal for references to the literature in case the answer is already known (although not to me, for which I apologise in advance).

I would also like to thank the members of the group "Mechanics, Quantisation and Geometry" for inviting me to Marseille, and helping to improve upon the first version of this note. However, any remaining mistakes or inaccuracies are entirely mine.

We study the problem by restricting elements of a coadjoint orbit to subalgebras. Such a restriction will not in general be a single orbit, but will be a union of orbits. In the resulting set of orbits for the subalgebra compactness of the original orbit may be lost since the new orbits need not, a priori, be closed. In stead we weaken compactness to boundedness which is preserved under restriction. So we study the apparently more general notion of bounded coadjoint orbits. At the last step we see that bounded orbits are necessarily compact, so we recover the desired situation.

2 Two technical lemmas

A class of Lie algebras where there are no unbounded coadjoint orbits are the abelian Lie algebras since there the coadjoint action is trivial so each orbit consists of a single point fixed under the whole group.

Definition 1 Say that a connected Lie group has property B⇒F if each bounded coadjoint orbit consists of a single fixed point.

Obviously, abelian Lie groups have property B⇒F.

Lemma 1 *Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$ and $r\colon \mathfrak{g}^* \to \mathfrak{h}^*$ be the restriction map. Suppose that $\mathfrak{h}$ has property B⇒F and $\mathcal{O}$ is a bounded orbit in $\mathfrak{g}^*$. Then $r(\mathcal{O})$ is a single point, a fixed point for the action of G on $\mathfrak{h}^*$ as automorphisms of $\mathfrak{h}$.*

Proof Let $H \subset G$ be the corresponding Lie groups and $\mathcal{O} \subset \mathfrak{g}^*$ a bounded coadjoint orbit of G. $r(\mathcal{O})$ is a bounded set in $\mathfrak{h}^*$ which breaks up into bounded H orbits. Since $\mathfrak{h}$

has property B⇒F, these must then be fixed points for H, so we have the basic result:

$$f \in \mathcal{O} \quad \Rightarrow \quad h.(f|_{\mathfrak{h}}) = f|_{\mathfrak{h}}, \quad \forall h \in H.$$

Since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, $h.(f|_{\mathfrak{h}}) = (h.f)|_{\mathfrak{h}}$ and so $h.f - f$ vanishes on $\mathfrak{h}$.

In particular $h.f - f$ vanishes on $[\mathfrak{h}, \mathfrak{g}]$ using again that $\mathfrak{h}$ is an ideal. Then $h_1.(h_2.f - f) = h_2.f - f$ for all h_1, h_2 in H. If we set $\varphi(h) = h.f - f$ then $\varphi(h_1h_2) = \varphi(h_1) + \varphi(h_2)$ so that $\varphi\colon H \to \mathfrak{g}^*$ is a homomorphism of Lie groups to the additive group of $\mathfrak{g}^*$. The image lies in $\mathcal{O} - f$ and so is bounded. But the only bounded subgroup of a vector space is the trivial subgroup $\{0\}$. Hence $\varphi(h) = 0$ for all h and thus $h.f = f$. It follows that f vanishes on $[\mathfrak{h}, \mathfrak{g}]$ and hence that $g.f - f$ vanishes on $\mathfrak{h}$ for all $g \in G$. Thus $r(\mathcal{O})$ is a single point. Since $\mathfrak{h}$ is an ideal, $(g.f)|_{\mathfrak{h}} = g.(f|_{\mathfrak{h}})$ and the rest of the statement follows. □

Lemma 2 *Let*

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{k} \to 0$$

be a short exact sequence of Lie algebras. If $\mathfrak{h}$ and $\mathfrak{k}$ have property B⇒F then so has $\mathfrak{g}$.

Proof We know from the previous Lemma that $r(f)$ is a single G-fixed point in $\mathfrak{h}^*$.

Pick $f_0 \in \mathcal{O}$ and consider the G orbit of $f - f_0$. Since f, $g.f$, $g.f_0$ all equal to f_0 on $\mathfrak{h}$, all points in the orbit of $f - f_0$ vanish on $\mathfrak{h}$ and so project to a K-orbit on $\mathfrak{k}^*$. Since $g.(f - f_0)$ lies in the bounded set $\mathcal{O} - \mathcal{O}$, we obtain a bounded orbit in $\mathfrak{k}^*$ and since $\mathfrak{k}$ has property B⇒F, this is a single K-fixed point. Hence $g.(f - f_0) = f - f_0$ for all $g \in G$ since G induces the K-action.

But $f = g'.f_0$ so $gg'.f_0 - g.f_0 = g'.f_0 - f_0$. Thus defining $\varphi(g) = g.f_0 - f_0$, we obtain a homomorphism $\varphi\colon G \to \mathfrak{g}^*$ whose image is bounded, and hence is zero. Thus $g.f_0 = f_0$ and hence $\mathcal{O}$ consists of a single fixed point. Thus G has property B⇒F. □

3 Solvable Lie Algebras

Theorem 1 *If $\mathfrak{g}$ is solvable then $\mathfrak{g}$ has property B⇒F.*

Proof A solvable Lie algebra can be built from successive extensions by abelian Lie algebras starting from an abelian Lie algebra, and, as we have already observed, abelian Lie algebras trivially have property B⇒F. Thus all solvable Lie algebras have property B⇒F. □

4 Semisimple Lie Algebras

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathcal{O} \subset \mathfrak{g}^*$ a coadjoint orbit. Then using the Killing form, $\mathcal{O}$ corresponds with a conjugacy class $\mathrm{Ad}\, G(\xi)$ in $\mathfrak{g}$ which is bounded if and only if $\mathcal{O}$ is. $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g}_i$, and ξ a sum of elements $\xi_i \in \mathfrak{g}_i$. $\mathcal{O}$ will be bounded if and only if all the conjugacy classes of the ξ_i are bounded. So in looking for bounded coadjoint orbits of semisimple Lie algebras we can assume $\mathfrak{g}$ is simple. We claim that a simple Lie algebra only has a non-trivial bounded conjugacy class when the Lie algebra is compact or equivalently that non-compact real simple Lie algebras have property B⇒F.

Lemma 3 *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\xi \in \mathfrak{g}$ have a bounded conjugacy class. If $\eta \in \mathfrak{g}$ has $\mathrm{ad}\,\eta$ semisimple with only real eigenvalues then $[\xi, \eta] = 0$.*

Proof Under the above assumptions, $\mathfrak{g}$ will decompose as a direct sum of eigenspaces $\mathfrak{g}_\lambda$ for $\mathrm{ad}\,\eta$. If $\xi = \sum_\lambda \xi_\lambda$ is the corresponding decomposition of ξ then

$$e^{t\,\mathrm{ad}\,\eta}\xi = \sum_\lambda e^{t\lambda}\xi_\lambda.$$

The right hand side can only be bounded if $\xi_\lambda = 0$ whenever $\lambda \neq 0$. Hence $[\eta, \xi] = 0$. □

Theorem 2 *If $\mathfrak{g}$ is a non-compact simple Lie algebra then $\mathfrak{g}$ has property B⇒F. In this case, the only fixed point is the origin.*

Proof A non-compact simple Lie algebra is generated by elements η for which $\mathrm{ad}\,\eta$ is semisimple with all eigenvalues real. To see this just take any Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. For $\mathfrak{g}$ non-compact $\mathfrak{p} \neq 0$. But $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ imply that $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a non-trivial ideal in $\mathfrak{g}$ so $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ since $\mathfrak{g}$ is simple. All elements η of $\mathfrak{p}$ are ad-semisimple with real eigenvalues. If ξ corresponds under the Killing form with an element of a bounded coadjoint orbit then we have $[\xi, \mathfrak{g}] = 0$ by Lemma 3 and hence $\xi = 0$. □

5 The General Case

Let $\mathfrak{g}$ be an arbitrary Lie algebra and $\mathfrak{r}$ its solvable radical. Then $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semi-simple. We can separate $\mathfrak{s}$ into compact and non-compact ideals $\mathfrak{s} = \mathfrak{s}_c + \mathfrak{s}_n$ and let $\mathfrak{g}_n$ be the

ideal of $\mathfrak{g}$ projecting onto $\mathfrak{s}_n$. Then $\mathfrak{r} \subset \mathfrak{g}_n$ with $\mathfrak{g}_n/\mathfrak{r} \cong \mathfrak{s}_n$; since both $\mathfrak{r}$ and $\mathfrak{s}_n$ have property B⇒F so does $\mathfrak{g}_n$ by Lemma 2. The quotient $\mathfrak{g}/\mathfrak{g}_n \cong \mathfrak{s}_c$ is compact semisimple. We shall show that a bounded coadjoint orbit $\mathcal{O}$ is essentially a coadjoint orbit of the compact semisimple Lie algebra $\mathfrak{s}_c$.

Let $\mathcal{O} \subset \mathfrak{g}$ be a bounded orbit. Then the restrictions $r(f)$ to $\mathfrak{g}_n$ of elements f of $\mathcal{O}$ are all equal to a single G-fixed point in $\mathfrak{g}_n^*$ by Lemma 1. Pick $f_0 \in \mathcal{O}$ and consider $f - f_0 \in \mathfrak{g}^*$. Its orbit $\mathcal{O}'$ will also be bounded and all the points in $\mathcal{O}'$ vanish on $\mathfrak{g}_n$. It follows that $\mathcal{O}'$ projects isomorphically to an orbit in the dual of $\mathfrak{g}/\mathfrak{g}_n$ and so is an orbit of a compact semisimple Lie group.

If we choose a Levi factor so that $\mathfrak{s}$ becomes a subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ is a semidirect product then $\mathfrak{g}_n = \mathfrak{r} + \mathfrak{s}_n$ and $\mathfrak{g} = \mathfrak{g}_n + \mathfrak{s}_c$. f_0 restricted to $\mathfrak{g}_n$ is independent of f_0 and is a G fixed element f_1 of $\mathfrak{g}_n^*$ which we can extend by 0 to $\mathfrak{g}$. Then $\mathcal{O} = f_1 + \mathcal{O}'$ where elements of $\mathcal{O}' \subset \mathfrak{s}_c$ are extended to $\mathfrak{g}$ as zero on $\mathfrak{g}_n$. In summary:

Theorem 3 *If $\mathcal{O}$ is a bounded coadjoint orbit of a Lie algebra, then $\mathcal{O}$ is closed, hence compact and has the form $\mathcal{O} = f_1 + \mathcal{O}_c$ where $\mathcal{O}_c$ is (the pull-back of) a coadjoint orbit of the compact semisimple quotient and f_1 is a G-fixed element of the dual of the non-compact part $\mathfrak{g}_n$.*